JOURNAL OF NATURAL GEOMETRY, 19(2001)93-120. 0963-2654-2001-19-6-93-120.

US$ 28.00. UK£ 14.00.

Conformal Background of Euclidean Trigonometry

Klaus Ruthenberg

Abstract. Essential parts of Euclidean trigonometry can be seen as trigonometry of conformal tetraglobes. Metrical basic structures of natural 3-dimensional space may be defined with the help of tetraglobes as first elements of this space, without using Euclidean basic concepts as 'straight line', 'length' and 'locality'.

Introduction

This paper is the second part of the work reported in [3] which for brevity will be denoted by 'Article I', and I.x will denote item numbered x in I, that is, in [3]. Measurement of angles and elementary angle functions were defined in Article I with the help of *real* cross ratios, the characteristic numbers of 2-circles (Definition I.6.1). Here we interpret a *complex* cross ratio as the characteristic number of a 4-circle, which leads to a conformal form of Euclidean trigonometry. Four points and the four circles through any three of these points produce a 4-circle or tetraglobe. The traditional form of Euclidean trigonometry defines the measurement of angles

2000 Mathematics Subject Classification: Primary: **30C35**; Secondary: **30E99**.

by ratios of sides in right-angled triangles and describes the relationship between lengths and angles in Euclidean triangles. This classical form of trigonometry **starts** with the Euclidean concept of '*lengths*' of '*straight lines*'. This article shows that a conformal form of Euclidean trigonometry can be constructed by **starting** with the conformal concept of *cross ratio* (*double ratio*). Essential parts of this construction can be done without using lengths of straight lines. Only at last, after the measurement of angles and the conformal form of trigonometry are defined, we may **add** the concept of length if we (physically) possess units of length. This 'diametrical' possibility to construct Euclidean trigonometry produces a new picture of our natural 3-dimensional space. This space (of our visual perception; of physics) can be seen as a conformal space with tetraglobes as first (basic) elements. The trigonometry of tetraglobes defines the (trigono)metrical basic structure of this space. But only if units of length are added (can be added) the natural space gets a complete Euclidean structure.

1. The parametric description of the cross ratio

I want to show that for the Euclidean triangle the sum of angles has a conformal substratum. We take three complex numbers

$$\begin{aligned} w_1 &= \sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot e^{i\alpha_1} \\ w_2 &= \sin^{-1}\alpha_1 \cdot \sin\alpha_3 \cdot e^{i\alpha_2} \\ w_3 &= \sin^{-1}\alpha_2 \cdot \sin\alpha_1 \cdot e^{i\alpha_3} \end{aligned} \tag{1.1}$$

with $0 < \alpha_k < \pi$ and define

Definition 1.1. *The three complex numbers w_k are called* ***representatives****, the three real numbers α_k the* ***arguments of a cycle*** $\{w_k\}$.

Lemma 1.1. *If the representatives of a cycle can be seen as the cyclic permutations of a cross ratio ((I.5.2) - (I.5.4)) the sum of the three arguments is π.*

Proof. If the w_k are the cyclic permutations of a cross ratio with (I.5.4) we have

$$\begin{aligned} w_1 \cdot w_2 \cdot w_3 &= -1 \\ e^{\boldsymbol{i} \cdot \Sigma \alpha_k} &= -1 \end{aligned}$$

and together with $\alpha_k > 0$

$$\sum \alpha_k = \pi. \tag{1.2}$$

□

Lemma 1.2. *If* $\sum \alpha_k = \pi$, *the three representatives* w_k *of a cycle are cyclic permutations of a cross ratio.*

Proof. With $\sum \alpha_k = \pi$ we get

$$\begin{aligned} &w_1(1 - w_3) = \\ &= \sin^{-1} \alpha_3 \cdot \sin \alpha_2 \cdot e^{\boldsymbol{i}\alpha_1}(1 - \sin^{-1} \alpha_2 \cdot \sin \alpha_1 \cdot e^{\boldsymbol{i}\alpha_3}) \\ &= \sin^{-1} \alpha_3 \cdot (\sin \alpha_2 \cdot e^{\boldsymbol{i}\alpha_1} - \sin \alpha_1 \cdot e^{\boldsymbol{i}(\alpha_1+\alpha_3)}) \\ &= \sin^{-1} \alpha_3 \cdot (\sin \alpha_2 \cdot e^{\boldsymbol{i}\alpha_1} + \sin \alpha_1 \cdot e^{-\boldsymbol{i}\alpha_2}) \\ &= \sin^{-1} \alpha_3 \cdot (\sin \alpha_2 \cdot \cos \alpha_1 + \boldsymbol{i} \cdot \sin \alpha_2 \cdot \sin \alpha_1 + \sin \alpha_1 \cdot \cos \alpha_2 - \boldsymbol{i} \cdot \sin \alpha_1 \cdot \sin \alpha_2) \\ &= \sin^{-1} \alpha_3 \cdot \sin(\alpha_1 + \alpha_2) \\ &= 1\,. \end{aligned}$$

The cyclic permutations w_1 and w_3 fulfil the first equation of (I.5.3). Also the second and third equation

$$\begin{aligned} w_3(1 - w_2) &= 1 \\ w_2(1 - w_1) &= 1 \end{aligned}$$

can be proved by replacing the indices. □

Because every non-real number w or its inverse w^{-1} can be written in the form

$$\sin^{-1} \alpha_3 \cdot \sin \alpha_2 \cdot e^{\boldsymbol{i}\alpha_1}$$

with $0 < \alpha_k < \pi$ and $\sum \alpha_k = \pi$ we get

Theorem 1.1. *The 2×3 different permutations of a non-real cross ratio can always be written in their parametric forms*

$$\begin{array}{ll} w_{+1} = \sin^{-1}(+\alpha_3) \cdot \sin(+\alpha_2) \cdot e^{i(+\alpha_1)} & w_{-1} = \sin^{-1}(-\alpha_2) \cdot \sin(-\alpha_3) \cdot e^{i(-\alpha_1)} \\ w_{+2} = \sin^{-1}(+\alpha_1) \cdot \sin(+\alpha_3) \cdot e^{i(+\alpha_2)} & w_{-2} = \sin^{-1}(-\alpha_3) \cdot \sin(-\alpha_1) \cdot e^{i(-\alpha_2)} \\ w_{+3} = \sin^{-1}(+\alpha_2) \cdot \sin(+\alpha_1) \cdot e^{i(+\alpha_3)} & w_{-3} = \sin^{-1}(-\alpha_1) \cdot \sin(-\alpha_2) \cdot e^{i(-\alpha_3)} \end{array} \quad (1.3)$$

with the help of 3 real parameters α_k, $0 < \alpha_k < \pi$, *and* $\sum \alpha_k = \pi$.

2. The conformal tetraglobe

We get a geometrical interpretation of the cross ratio parametric description with the help of a conformal basic figure:

Definition 2.1. *Four points and the four conformal circles through any three of these four points produce a* 4-***circle*** *or* ***tetraglobe (tetra)****.*

Remarks

1. It is helpful to see four points on one circle as a degenerate 4-circle.
2. 'Tetra' remembers that a 4-circle possesses four dual elements:

   - Three points are on every circle.

   - Three circles go through every point.
3. Tetra-'globe' remembers that the four points of every (not degenerate) 4-circle define a sphere in the 3-dimensional conformal space; a sphere which 'carries' the 4-circle. This conformal sphere is a 'stage' where the tetra can be conformally moved.
4. A tetraglobe possesses six 2-circles (Section 6 in [3]).

Definition 2.2. *Two* 2*-circles of a tetra without common corners are* 2*-circles in* ***conformal position****.*

I use the expression '2-circles in conformal position' as Germans use the set phrase 'Figuren in projektiver Lage (figures in projective position)'.

Two 2-circles in conformal position 'generate' a 4-circle. Conversely a 4-circle can be seen as two 2-circles in conformal position in three ways.

It is often helpful to discuss a conformal tetraglobe in a special, 'Euclidean' location.

Definition 2.3. *A tetraglobe is in an **Euclidean location** if one of its corners is the point $z = \infty$.*

Remark

1. If we use the traditional Gauss/Argand plane to illustrate a 4-circle in Euclidean location, each tetraglobe looks like an Euclidean triangle together with its circumcircle. But in conformal geometry the point $z = \infty$ is not a non proper point; the circles through $z = \infty$ appear, if we use this illustration, like Euclidean straight lines but they are conformal circles; the circle through the three corners of such an 'Euclidean triangle' is a conformal circle without centre and radius.
2. It is often helpful to draw conformal circles in the Gauss plane with a ruler but often a 4-circle drawn on a Riemann sphere gives a better picture of a tetraglobe. For example this model demonstrates more directly the symmetry of every tetraglobe.

Theorem 2.1. *Each conformal tetra is a geometrical entity with a symmetry group of four elements.*

Proof. Always a conformal mapping exists which transforms a 4-circle with the four points z_k into a special location so that

$$z_1 = 0, \quad z_2 = 1, \quad z_3 = \infty, \quad z_4 = a \,, \tag{2.1}$$

where the complex number a is determined by the special shape of the tetra.

With

$$\begin{aligned} z_1 = 0 \quad &\longrightarrow \quad z_2 = 1 \\ z_2 = 1 \quad &\longrightarrow \quad z_1 = 0 \\ z_3 = \infty \quad &\longrightarrow \quad z_4 = a \end{aligned} \tag{2.2}$$

we define the mapping

$$z \longrightarrow z' : \quad zz' - a(z + z') + a = 0 \;, \tag{2.3}$$

which changes z_1, z_2 into z_2, z_1 and z_3, z_4 into z_4, z_3.

In pairing other points in the same way we get the tetra's symmetry group with the four elements

$$\begin{aligned} z' - z &= 0 \\ z'z - a &= 0 \\ z'z - (z' + z) + a &= 0 \\ z'z - a(z' + z) + a &= 0. \end{aligned} \tag{2.4}$$

□

Theorem 2.2. *If two 2-circles of a tetraglobe lie in a conformal position their angles are equal.*

Proof. With the three non-identical symmetry mappings every pair of 2-circles in conformal position interchange their positions. □

Definition 2.4. *Every three points of a tetraglobe define a* ***conformal triangle****; the fourth point is the* ***pole*** *of this triangle. Each point pair of a conformal triangle defines a* ***side*** *of this triangle. Each pair of the sides, arcs on the tetraglobe's circles, defines an* ***angle of the triangle***

Theorem 2.3. *The four conformal triangles of a 4-circle are conformally equivalent; every triangle of a tetra has three angles of the same magnitude.*

Proof. Each conformal triangle of a tetra can be transformed into every other one with a suitable element of the tetra's symmetry group. □

If we identify an angle with its magnitude and if we neglect the difference of 'angle', 'apex angle' and 'adjacent angle' (Definition I.6.2) one may say that at most three different angles exist in a tetraglobe. These three angles appear in each triangle of the tetraglobe as angles of this triangle. Because in this sense a tetra possesses at most three different angles we also call such a tetraglobe a **3-angle**. This word emphasises both the relationship of a conformal tetraglobe with an Euclidean triangle and the difference between an Euclidean triangle and a conformal 3-angle.

3. The characteristic number of a tetraglobe

Four points z_k define exactly one conformal, complex cross ratio and exactly one tetraglobe. In $\mathbb{M}$ and in ${}^{r}\mathbb{M}^{\lambda}$ (with even λ) this number w is invariantly bound with the tetra.

Definition 3.1. *The complex cross ratio of the four tetraglobe points is the* ***characteristic number*** *of the* 4*-circle.*

Therefore the characteristic number of a tetraglobe has 2×3 'permutations', namely the 2×3 variations (I.5.2) of the cross ratio (I.5.1).

Definition 3.2. *The three real parameters α_k of the parametric representation (1.3) of a cross ratio are the* ***magnitudes*** *of the three angles of a conformal* 3*-angle (tetraglobe).*

This definition suggests itself because $\sum \alpha_k = \pi$ (Theorem 1.1); and in a Gauss plane a tetraglobe in Euclidean location appears as an Euclidean triangle, which also has this angle sum. That this definition is also admissible I prove in the following by comparing the measuring of angles by a tetraglobe with the measuring of angles in Euclidean trigonometry and with the help of a measuring circle (Definition I.7.1). If a cross ratio is seen as the characteristic number of a tetra, often qualities of a cross ratio can be seen as attributes of a tetraglobe, for instance:

A cross ratio possesses only 2×3 different permutations, because the three angles with the three positive magnitudes $+\alpha_k$ and the three negative magnitudes $-\alpha_k$ (Theorem 1.1 with (1.3)) have only 2×3 permutations.

The characteristic number w of a tetraglobe changes into the conjugate number $\overline{w}$ if we map this geometrical figure by an **odd** conformal transformation (Definition I.1.2). What does this changing mean geometrically? If the characteristic number of a tetraglobe is non-real (if the tetraglobe is not degenerate) always a pair of tetraglobes exists which do not differ in the magnitudes of their angles, they are mirror images of each other. An odd conformal transformation maps a right tetra into a left one.

4. Directions, positions and orientations of tetraglobes

A tetraglobe in $\mathbb{C}$ can be given two directions, two positions and two orientations.

Definition 4.1. *The two* ***directions in a tetraglobe*** *are defined with the help of the permutations of its three angles* $\alpha_k > 0$*:*

$$\textit{\textbf{For(ward) direction}} \quad d_f := \{(\alpha_1, \alpha_2, \alpha_3), (\alpha_2, \alpha_3, \alpha_1), (\alpha_3, \alpha_1, \alpha_2)\}$$
$$\textit{\textbf{Back(ward) direction}} \quad d_b := \{(\alpha_2, \alpha_1, \alpha_3), (\alpha_3, \alpha_2, \alpha_1), (\alpha_1, \alpha_3, \alpha_2)\}$$

It is convenient to see $+\alpha_k > 0$ as the measuring numbers of the three tetra angles if this tetraglobe possesses a forward direction; and to see $-\alpha_k < 0$ as the measuring numbers if this tetra is backward directed.

Definition 4.2. *The two* ***positions of a tetraglobe*** *(in its 2-dimensional stage $\mathbb{C}$) are defined with the help of the unit* $\boldsymbol{i}$*:*

$$\textbf{\textit{Out(ward)position}} \textit{ by } \quad +\boldsymbol{i}$$

$$\textbf{\textit{In(ward)position}} \quad \textit{by} \quad -\boldsymbol{i}$$

Definition 4.3. *The two* ***orientations or spin directions*** *of a tetraglobe are defined both by its directions and its positions:*

$$\textbf{\textit{Right orientation}} \quad s_r := \{(+\boldsymbol{i},\, d_f),\, (-\boldsymbol{i},\, d_b)\}$$

$$\textbf{\textit{Left orientation}} \quad s_l := \{(+\boldsymbol{i},\, d_b),\, (-\boldsymbol{i},\, d_f)\}$$

Remarks

1. The terms 'for - back', 'out - in', 'right - left' may be used with the opposite sense, too. The chosen words are arbitrary. For example a tetra with an 'outward position' may also be seen as a tetra with an 'inward position' if also the meaning of 'inward position' is changed in the same sense.

2. 'Orientation' and 'spin direction' are more suitable words to describe these two possible states of a tetraglobe than for instance 'screw sense' or 'sense of rotation'. A conformal tetra possesses two 'spin directions' but not a centre of rotation. ('Circulation round the centre of a circle' can be defined in Euclidean but not generally in conformal geometry).

3. Mathematics traditionally sees the **imaginary** unit as a point (or unit vector) **in** the Gauss/Argand plane. I use as **complex** unit $\boldsymbol{i}$, with $\boldsymbol{i}^2 = -1$, a

(quaternionic) unit vector **orthogonal** to the tetraglobe and its 'stage' $\mathbb{C}$ [2]. $\boldsymbol{i}$ is helpful to see a tetraglobe together with its stage $\mathbb{C}$ as a structure 'swimming' with a fixed position in the natural 3-dimensional space of our visual perception. If $\mathbb{C}$ is seen as such element in the 3-dimensional conformal space the quaternionic unit $\boldsymbol{i}$ describes not only the in- and outward position of a tetra but also the position of this conformal 3-angle (and its $\mathbb{C}$) in our 3-dimensional space. Only if in the Gauss plane (or Riemann sphere) a unit of length exists, the complex $\boldsymbol{i}$ can be identified with the imaginary unit in the stage $\mathbb{C}$.

5. Magnitudes of angles and trigonometric functions defined by orthogonal tetras

Definition 5.1. *A tetraglobe is* ***orthogonal*** *or* ***right-angled*** *if one of the three parameters α_k, which describe the characteristic number of this 4-circle, is $\pi/2$.*

An orthogonal tetra can be used to measure an angle φ. It is expedient to substitute 'w' by 'v' if the characteristic numbers of right angled tetras are used as *angle measuring tetras.* If

$$\alpha_1 = \pi/2\ ,\ \alpha_2 = \varphi\ ,\ \alpha_3 = \pi/2 - \varphi \tag{5.1}$$

the characteristic number v_1 of the orthogonal measuring tetra with these angles is

$$v_1 = \sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot e^{\boldsymbol{i}\pi/2} = \sin\varphi \cdot \sin^{-1}(\pi/2 - \varphi) \cdot e^{\boldsymbol{i}\pi/2} = \boldsymbol{i} \cdot \tan\varphi\ . \tag{5.2}$$

A measuring right-angled 3-angle with the angles (5.1) has these 2×3 permutations $v_{\pm k}$ of its characteristic number:

$$\begin{array}{ll} v_{+1} = \boldsymbol{i} \cdot \tan\varphi & v_{-1} = -\boldsymbol{i} \cdot \tan^{-1}\varphi \\ v_{+2} = \cos\varphi \cdot (\cos\varphi + \boldsymbol{i} \cdot \sin\varphi) & v_{-2} = \cos^{-1}\varphi \cdot (\cos\varphi - \boldsymbol{i} \cdot \sin\varphi) \\ v_{+3} = \sin^{-1}\varphi \cdot (\sin\varphi + \boldsymbol{i} \cdot \cos\varphi) & v_{-3} = \sin\varphi \cdot (\sin\varphi - \boldsymbol{i} \cdot \cos\varphi)\ , \end{array} \tag{5.3}$$

so that

$$\begin{array}{ll} v_{+1}\overline{v}_{+1} = \tan^{+2}\varphi & v_{-1}\overline{v}_{-1} = \tan^{-2}\varphi \\ v_{+2}\overline{v}_{+2} = \cos^{+2}\varphi & v_{-2}\overline{v}_{-2} = \cos^{-2}\varphi \\ v_{+3}\overline{v}_{+3} = \sin^{-2}\varphi & v_{-3}\overline{v}_{-3} = \sin^{+2}\varphi\ . \end{array} \tag{5.4}$$

The 2×3 permutations $v_{\pm k}$, the permutations of the characteristic number of the measuring tetraglobe, can be used to define the 2×3 trigonometrical functions:

Definition 5.2.

$$\begin{array}{rr} \tan\varphi := (v_{+1}\overline{v}_{+1})^{1/2} & \tan^{-1}\varphi := (v_{-1}\overline{v}_{-1})^{1/2} \\ \cos\varphi := (v_{+2}\overline{v}_{+2})^{1/2} & \cos^{-1}\varphi := (v_{-2}\overline{v}_{-2})^{1/2} \\ \sin^{-1}\varphi := (v_{+3}\overline{v}_{+3})^{1/2} & \sin\varphi := (v_{-3}\overline{v}_{-3})^{1/2} \end{array} \tag{5.5}$$

We now have a new definition of classical trigonometrical functions with the help of the conformal cross ratios without using any Euclidean structure.

In Article I [3] only real cross ratios were used for this definition; here we use non-real cross ratios which are the characteristic numbers of ortho-tetras.

6. The conformal Pythagorean structure

In the following we define a structure which can help to discuss the connection of conformal trigonometry to known forms of angle measurement.

Given a 2-circle UH (with the circles U and H and the corners z and z') together with a circle C which measures UH. The measuring points are z_1 and z_3 on U; z_2 and z_4 on H.

Definition 6.1. *UH together with C and the six points z, z', z_k generate a* ***conformal Pythagorean structure****.*

At first we use this structure to discuss the connection between the measurement of angles with a measuring circle (compare Definition I.8.1 and Equation (I.7.8)) and with an orthogonal tetraglobe.

The Pythagorean structure may be completed by the circles $A = z_3 z z_2$ and $A' = z_3 z' z_2$.

We especially discuss the angle in UH with the measuring points z_1 and z_4. This angle and its magnitude is called ϕ. The characteristic number, measuring this angle with the help of C, is $u = (z_1 z_2 z_3 z_4)$ (Also here we substitute w of Definition I.7.2 by a new letter u because the cross ratio u is especially a measuring number).

So with (I.7.8) it follows

$$u = -\tan^2(\phi/2) \ . \tag{6.1}$$

To compare the angle measurement by C and the measurement by a right-angled tetra both the triangle $z z_3 z_4$ (with the pole z_2) and the triangle $z' z_3 z_4$ (with the pole z_2) are used.

2-circle UH and 2-circle AA' are 2-circles in conformal position in the 4-circle $z z' z_2 z_3$. Therefore the magnitude of the angle ϕ appears also as magnitude of the angle $z z_3 z'$ (with the arc $z z_3$ on A and the arc $z' z_3$ on A'). z and z' are symmetrical points in relation to the symmetry C; z_3 and z_4 are fixed points of this symmetry. So we can use not only C but also the orthogonal triangle $z z_3 z_4$ (with the angle $\varphi = z z_3 z_4$) or the orthogonal triangle $z' z_3 z_4$ (with the angle $\varphi' = z' z_3 z_4$) for measuring of $\phi = 2\varphi = 2\varphi'$ because the symmetrical angles φ, φ' are equal, $\varphi = \varphi'$, and $\phi = \varphi + \varphi'$.

The characteristic numbers of the orthogonal tetraglobes $z z_2 z_3 z_4$ and $z' z_2 z_3 z_4$ are conjugates because both are in a symmetric position,

$$v_1 = +\boldsymbol{i} \cdot \tan\varphi \qquad \overline{v}_1 = -\boldsymbol{i} \cdot \tan\varphi' \ . \tag{6.2}$$

And because $\varphi = \varphi' = \phi/2$:

$$v_1 = +\boldsymbol{i} \cdot \tan\phi/2 \qquad \overline{v}_1 = -\boldsymbol{i} \cdot \tan\phi/2 \ . \tag{6.3}$$

This together with (6.1) leads to

$$v_1^2 = \overline{v}_1^2 = u \,. \tag{6.4}$$

(Compare (I.9.6))

Theorem 6.1. *If an angle ϕ is measured by the real characteristic (measuring) number u and also by the imaginary measuring number v_1 (or $\overline{v}_1$) the equation (6.4) describes the connection between these different measuring numbers.*

The two conformal possibilities to measure angles are compatible.

The Pythagorean figure shows the connection between the measuring with the help of an ortho-circle and with the help of an ortho-tetra, but also an essential difference: With ortho-circles measured angles are **not directed**. But if an angle is part of an ortho-tetra the two possible directions of this tetraglobe can be used to give also an angle a direction.

Theorem 6.2. *The 2×3 permutations $v_{\pm k}$ of an ortho-tetra for measuring an angle $\varphi = \phi/2$ and the 2×3 permutations $u_{\pm k}$ of an ortho-circle for measuring this angle $\varphi = \phi/2$ are connected by*

$$\begin{array}{ll} u_{+1} = -v_{+1}\overline{v}_{+1} = v_1^2 & u_{-1} = -v_{-1}\overline{v}_{-1} = \overline{v}_{-1}^2 \\ u_{+2} = v_{+2}\overline{v}_{+2} & u_{-2} = v_{-2}\overline{v}_{-2} \\ u_{+3} = v_{+3}\overline{v}_{+3} & u_{-3} = v_{-3}\overline{v}_{-3} \end{array} \tag{6.5}$$

I call these equations the **conformal Pythagorean equations**. They are proved by comparing (I.9.2) and (5.4).

7. The Pythagorean structure in Euclidean location

In general location the Pythagorean structure was used to discuss the connection between the angle measuring with a real and with a non-real cross ratio.

To compare conformal and Euclidean structures we use the Pythagorean figure in a special, Euclidean location. We chose $z_2 = \infty$ (see Section 6), $z_4 = 0$ and describe C by the equation $z = \overline{z}$. So the other points are simple $z_3 = p$, $z_1 = -q$ and $z = \boldsymbol{i} \cdot h$ with real p, q, h.

Till now we have not left the conformal standpoint, we have only chosen a special location of the conformal structure to make easy the reckoning. Together with C, A, defined in Section 6, we use the cocircle $B = zz_1z_2$, too. If we also use a Gauss plane to illustrate the Pythagorean structure the tetra $z_1z_2z_3z$ appears as the conformal triangle z_1z_3z represented by the three Euclidean straight lines A, B, C.

Theorem 7.1. *A Pythagorean figure with the lines A, B, C, H possesses three conformally equal tetras*

$$_aw = {_bw} = {_cw}$$

with

$$_aw := (zz_2z_3z_4), \qquad _bw := (z_1z_2zz_4), \qquad _c\overline{w} := (z_1z_2z_3z)\ . \tag{7.1}$$

Proof. Using our special location we have

$$_aw = \boldsymbol{i} \cdot h/p \tag{7.2}$$

$$_bw = \boldsymbol{i} \cdot q/h \tag{7.3}$$

$$_cw = -(pq - h^2 - \boldsymbol{i} \cdot h(p+q))(h^2 + p^2)^{-1}\ . \tag{7.4}$$

Together with

$$u_{+1} = (z_1z_2z_3z_4) = -q/p \tag{7.5}$$

$$v_{+1} = {_aw} = \boldsymbol{i} \cdot h/p \tag{7.6}$$

and because

$$u_{+1} = -v_{+1}\overline{v}_{+1} = v_{+1}^2 \quad (6.5)$$

it follows

$$h^2 = p \cdot q\ . \tag{7.7}$$

If this is used in (7.3) and (7.4) we get

$$ {}_b w = \boldsymbol{i} \cdot q/h = \boldsymbol{i} \cdot h/q = {}_a w \tag{7.8} $$

and

$$ \begin{aligned} {}_c w &= -(pq - h^2 - \boldsymbol{i} \cdot h(p+q)))(p^2+h^2)^{-1} \\ &= \boldsymbol{i} \cdot h(p+q))(p^2+h^2)^{-1} = \boldsymbol{i} \cdot h(p+q)(p^2+pq)^{-1} = \boldsymbol{i} \cdot h/p = {}_a w \ . \end{aligned} \tag{7.9} $$

So

$$ {}_a w = {}_b w = {}_c w $$

and Theorem 7.1 is proved. □

In Euclidean location the three tetras with equal characteristic numbers are represented by the three orthogonal triangles with the hypotenuses a, b, c on the lines A, B, C. Therefore Theorem 7.1 is the conformal expression for the Euclidean theorem that every orthogonal triangle together with its height can be seen as a system of three similar triangles.

In discussing the equation (7.7) we can go on to compare the 'Pythagorean' structure of an Euclidean right-angled triangle and the structure of our conformal Pythagorean figure. In conformal geometry we have only used h, p, q as real numbers to fix the location of points. But in Euclidean geometry h, p, q can be seen as lengths of the height and of distances on the hypotenuse $c = p + q$. In Germany we say: $h^2 = p \cdot q$ is an element of a set of four 'Pythagorean theorems'

$$ h^2 = p \cdot q \ , \quad c \cdot p = a^2 \ , \quad c \cdot q = b^2 \ , \quad c^2 = a^2 + b^2 \ . \tag{7.10} $$

In Euclidean geometry all Pythagorean theorems can be deduced by using the similarity of the three triangles with the hypotenuses a, b, c. So our **Theorems 7.1 and 6.2 describe the conformal background of these four Pythagorean theorems.**

A more detailed discussion of those structures of an Euclidean right-angled triangle that can also be seen as structures of the conformal Pythagorean figure is not the subject of this Section. But I wanted to emphasise that the conformal Pythagorean formulae (6.5) of Theorem 6.2 and especially the conformal equation

$$u_1 = v_1^2 \tag{7.11}$$

has in

$$q/p = (h/p)^2 \tag{7.12}$$

its Euclidean equivalent. (More directly as $p \cdot q = h^2$ the formula (7.12) simulates the form of (7.11) and expresses which ratios are related). The Euclidean theorem starts with lengths and ratios of such lengths, the conformal theorem is based on cross ratios only.

The Equations (6.5) does not show its relation with the measurement of angles. Also $p \cdot q = h^2$ does not show the implicit connection to the Euclidean form of angle measurement. Particularly we have answered the following question: Where does the conformal double possibility of measuring angles with real and complex cross ratios get their expression at the Euclidean level?

The conformal Pythagorean structure in its Euclidean location shows directly with (7.6) that **the conformal and the Euclidean measurement of angles are compatible** because in this location

$$\tan\varphi := v_1/\boldsymbol{i} = h/p\,. \tag{7.13}$$

Also the different starting points can be seen: h, p are sides of a right-angled Euclidean triangle, v_1 is the measuring number of an orthogonal tetra. This conformal characteristic number is defined without using Euclidean lengths.

The measurements of angles with the help of Euclidean triangles or with conformal ortho-circles and ortho-tetras are not only compatible but Euclidean geometry and conformal geometry define an equivalent measurement of angles by using different starting points. The two ways to construct trigonometry are different but 'diametrical': Euclidean trigonometry starts with right-angled Euclidean triangles, uses lengths of sides in these triangles and defines trigonometrical functions (angle measurement) by ratios of these sides. Not using 'straight lines' and 'lengths' of these lines conformal trigonometry starts with 2^0-circles which constitute angles in 2^1-circles. Tetras (2^2-circles) are pairs of 2-circles in conformal positions. Angle parameters describe the shape of tetras with the help of complex characteristic numbers. And such characteristic numbers can be used to measure angles and to define trigonometric functions.

The question, which parts of the Euclidean theory of right-angled triangles cannot be sublimated by a theory of conformal Pythagorean structure, is interesting; especially, which role Pythagoras' theorem $c^2 = a^2 + b^2$ plays in tetra structures if ∞ is not generally fixed. We discuss this question in Section 10.

8. Families of tetra numbers

Definition 8.1. *The 2×3 permutations of a tetraglobe's characteristic number are a* ***family of tetra numbers****.*

Because the members of such family are the permutations of the same cross ratio these six numbers $w_{\pm k}$ are bound by the typical equations (I.5.2) - (I.5.6). For example the unit $\boldsymbol{i}$ is member of the family

$$\begin{array}{ll} w_{+1} = \boldsymbol{i} & w_{-1} = -\boldsymbol{i} \\ w_{+2} = \frac{1}{2}(1+\boldsymbol{i}) & w_{-2} = 1 - \boldsymbol{i} \\ w_{+3} = 1 + \boldsymbol{i} & w_{-3} = \frac{1}{2}(1 - \boldsymbol{i}). \end{array} \tag{8.1}$$

This family of tetra numbers is generated by the tetraglobe with the three angles

$$\alpha_1 = \pi/2, \quad \alpha_2 = \pi/4, \quad \alpha_3 = \pi/4\ . \tag{8.2}$$

Three and only three families exist with less than 6 (different) numbers, namely the two 'real' families

$$\begin{array}{ll} w_{+1} = w_{-1} = -1 & w_{+1} = w_{-3} = 0 \\ w_{+2} = w_{-3} = \frac{1}{2} & w_{+2} = w_{-2} = +1 \\ w_{+3} = w_{-2} = 2 & w_{+3} = w_{-1} = \infty \end{array} \tag{8.3}$$

and one 'complex' family

$$\begin{aligned} w_{+1} = w_{+2} = w_{+3} &= \tfrac{1}{2}(1 + i\sqrt{3}) \\ w_{-1} = w_{-2} = w_{-3} &= \tfrac{1}{2}(1 - i\sqrt{3}). \end{aligned} \tag{8.4}$$

This family (8.4) with only two different elements is generated by the especially symmetrical tetraglobe with 3 equal angles $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$.

Remark

I see tetraglobes 'swimming' in the natural space, carried by its conformal sphere $\mathbb{C}_i$. Every non-degenerate tetraglobe can be seen as a coordinate system of its 'individual 2-dimensional world' $\mathbb{C}_i$. This world gets an *individual* but *complete* Euclidean structure if the fourth point of its coordinate tetra is seen as an absolute point z_{abs}.

I see a remarkable quality of such individual 2-dimensional tetra-world: It possesses an internal trigonometry and the spin of its coordinate system describes its external position in the 3-dimensional conformal space *but a diameter of this conformal globe is not defined.* We may regard such conformal sphere $\mathbb{C}_i$ and every tetra in this globe as small ('point') or as big ('plane') as we like. But because lengths are not defined, concepts such as 'small' and 'big' have no exact (objective) meaning. Both models (point, plane) may be interchanged just as physicists change between 'particle' and 'wave element'. My particle model of $\mathbb{C}_i$ and its tetraglobes is a pure mathematical model. As it is helpful to regard physical particles as conformal tetraglobes the especially symmetric tetra (8.4) may turn out to be particular interesting.

9. The sine law of a tetraglobe

Euclidean trigonometry defines trigonometric functions by ratios of sides in *right-angled* triangles. Conformal trigonometry defines such functions with the help of characteristic numbers (cross ratios) of *ortho*-circles and *ortho*-tetras. Euclidean trigonometry discusses also the relation of angles and sides in scalene triangles. In the following we discuss how conformal trigonometry may describe the relation of angles and sides in *general* tetras (and its triangles), also.

Every tetraglobe possesses 2×3 complex invariants $w_{\pm\kappa}$ and also 2 triples of real invariants $w_\kappa \overline{w}_\kappa$, $w_{-\kappa}\overline{w}_{-\kappa}$. In the following we interpret the meaning of these real triples invariantly bound with every tetraglobe.

Looking at the situation in Euclidean geometry I define:

Definition 9.1. $(w_\kappa\, \overline{w}_\kappa)^{1/2}$ *are the* ***ratios of*** *3* ***sides*** a_1, a_2, a_3 *existing in every tetraglobe.*

With this definition and with (1.3) of Theorem 1.1 it follows for such *ratios of sides* $(a_1/a_2), (a_2/a_3), (a_3/a_1)$:

$$(a_1/a_2) = \sin\alpha_1/\sin\alpha_2, \quad (a_2/a_3) = \sin\alpha_2/\sin\alpha_3, \quad (a_3/a_1) = \sin\alpha_3/\sin\alpha_1\ . \tag{9.2}$$

Definition 9.2. *Equations (9.2) describe the conformal* ***sine law of a tetraglobe****.*

It is important to see the difference of this conformal law and the Euclidean one. Equations (9.2) may formally also be written with help of a proportionality factor 2ρ:

$$a_k = 2\rho \cdot \sin\alpha_k\ . \tag{9.3}$$

However in conformal geometry the *length of a side* a_k and the factor ρ does not have an objective (invariant) meaning. Euclidean geometry is characterised by an other situation: Here every a_k can be seen as (the length of) a side and ρ can be seen as radius of a circumcircle, more exactly as length of this radius.

RUTHENBERG

If we draw a tetra in Euclidean location, using a Gauss plane, the tetra is also represented by an Euclidean triangle together with its circumcircle. But one must remember that this Euclidean triangle is only an illustration of a conformal 3-angle. The circumcircle is only a konformkreis without centre and radius. In this special location we may - for illustration only - use a pair of compasses to draw this konformkreis as a circumcircle. But if we do not leave the conformal standpoint we do not have a radius ρ of this circle and a length of a side in the triangle.

Remarks

1. In mathematics one writes a number in the polar coordinate form $z = \rho \cdot e^{i\varphi}$.

   We can also use this representation of a complex number z by such a pair (ρ, φ) for describing conformal structures if we do not geometrically interpret such coordinates as elements of Euclidean geometry. Only traditionally we interpret these polar coordinates with the help of Euclidean geometry: ρ is the *length of a radius*, φ *measures the rotation* about an origin. Conformal trigonometry of tetraglobes leads to *pure angle coordinates* of a complex number
$$z = \sin^{-1} \alpha_3 \cdot \sin \alpha_2 \cdot \exp(i\alpha_1), \quad \Sigma\alpha_\kappa = \pi \ .$$

2. Einstein reflected the invariant theoretical situation of metrical basic numbers. His result was: Physics only possesses a unit of time and mass in relation to an individual coordinate system. Therefore I compare the situation of a physicist sitting in a coordinate system of special relativity with the situation of an observer who is sitting on a conformal tetraglobe. This person may interpret a fourth point of the tetraglobe as the absolute point ∞ and use an individual unit ρ_0 of lengths. But this observer has a problem: The length unit ρ_0 cannot generally be compared with an individual length unit ρ_0' in another tetraglobe.

3. The last remark may help to understand the essential difference of the sine law in Euclidean and in conformal geometry. It may also suggest the question

if conformal geometry can help to analyse the axioms and conditions which determine a coordinate system of special relativity. Einstein gave a new interpretation of the Lorentz transformations which allows one to compare the basic units of time and mass in two inertial systems. But has theoretical physics constructed and discussed a **model** of 'inertial systems'? I think a tetraglobe as a model of a physical coordinate system. This system has a defined structure. But this structure is on the conformal level defined without using 'straight lines', 'length' and '(inertial) movements' on such lines.

10. On Pythagoras' theorem

The equation

$$\cos^2\varphi + \sin^2\varphi = 1 \tag{10.1}$$

is traditionally seen as the trigonometric form of Pythagoras' theorem. But in [3] this equation was proved as an equation between angle functions without any relation to a (conformal or Euclidean) orthogonal triangle (cf. [I.9.4]).

Euclid does not use angle functions to formulate Pythagoras' theorem.

The conformal analogy of Pythagoras' theorem is (for example)

$$v_{+2}\overline{v}_{+2} + v_{-3}\overline{v}_{-3} = 1 \tag{10.2}$$

which follows from (6.5) together with (I.5.6). In this formula related to a right-angled tetraglobe angle functions too do not appear.

This equation, which describes the relation between two cross ratios in any orthogonal tetra, can be written in a more usual form if the (conformal) concept of '*ratios of sides* (in a tetraglobe)' is used:

$$(a/c)^2 + (b/c)^2 = 1 \tag{10.3}$$

If and only if the right-angled tetraglobe possesses an (individual) unit of length, this conformal Pythagoras' equation can be transformed in the classical Euclidean form

$$a^2 + b^2 = c^2 \,. \tag{10.4}$$

In developing elementary Euclidean geometry as a conformal invariant theory I could expose the conformal fundamentals of Euclidean sinus law and of Pythagoras' theorem. If and only if the individual geometry of a tetraglobe possesses a unit length not only *ratios* of triangle sides but also *lengths* of such sides can be measured; sine law and Pythagoras' theorem can be formulated in the usual Euclidean form with the help of such unit length.

In revealing the conformal background of Euclidean geometry this article also could prove that some fundamental Euclidean theorems are essentially conformal ones:

- Euclidean angles with their apexes on an Euclidean circle and subtended by the same arc of this circle are equal because these angles are in a *conformal position* to the same angle.

- Euclidean triangles possess the angle sum π because a conformal tetraglobe possesses this angle sum.

In every Riemannian differential geometry Pythagoras' theorem is still valid in the local spaces. Pythagoras' theorem is generally not valid in the conformal space. The question is how Riemann's way to generalise geometry can be changed for getting a differential geometrical form of conformal geometry. This question is not a new one. To generalise conformal geometry Cartan [1] and Schouten [4] went a non-Riemannian way which was used also by Thomas [5], Veblen [6] and Yano [8]. In this connection Weyl's modification of Riemannian geometry also has to be seen [7]. But can mathematics give the very special structure of conformal tetraglobe geometry a non-linear (non-homogeneous, non-integrable displacement) form only by using the basic conceptions (and formalisms) of Cartan, Levi-Civita, Ricci [1,4] ?

In holding onto the Riemannian idea of *local tangential* spaces we may miss the right way to build the differential geometrical form of conformal tetra geometry. Is it sufficient to substitute the Euclidean group in the tangent spaces by a centro-conformal group (Cartan, Schouten) (or to suppose that a metrical tensor g_{ik} possesses an undefined factor (Weyl)) to give the conformal geometry of tetraglobes a non-linear form?

A repeatedly asked question in which form conformal geometry may be generalised in the sense of differential geometry (and function theory) has to give a new answer in such a way that

- only conformal circles exist, not Euclidean straight lines and Euclidean circles,

- angles and magnitudes of angles are defined without using the concepts of straight lines and lengths,

- tetraglobes are first elements of our space, not points.

How can a non-integrable displacement (Übertragung, connection) be defined between *individual* tetraglobes, not between *local* tangent spaces?

Riemann's generalisation from **2** not only to 3 but to **n** dimensions was a natural step. A conformal generalisation is only consistent with the specific shape of conformal tetra geometry if the characteristic restriction of this geometry to (2 and) **3** dimensions is an essential part of this generalisation, too. Classical complex analysis is not only by chance restricted to the two dimensions of normal complex numbers. I want to see the shape of this classical complex calculus generalised to three dimensions in such a way that everybody can understand the theory as the differential geometrical generalisation of conformal tetra geometry.

11. On angle units

Trigonometry defines measurement of angles but not the unit of angle measuring.

Mathematics and physics do not possess a natural length unit but they possess a *natural angle unit.*

Do we not possess a natural number to describe the natural angle unit? For in the last 2000 years scientists have used several angle units to describe the natural angle unit, for example the numbers

$$4, \quad 360, \quad 2\pi \ .$$

Is it optimal to use the **length** 2π of an **Euclidean unit circle** to describe the magnitude of angles in conformal tetra structures which generally do not possess a length unit? And why do we not use a **unit** to describe the natural angle **unit**?

If we want to use a unit to describe the natural angle unit I see two possibilities: We can use the real unit '1' or the imaginary unit 'i'.

Many formulae in physics suggest that the Planck constant h should be replaced by $h/2\pi$. Why not - in this connection - substitute the angle unit number 2π by 1?

The substitution $2\pi \longrightarrow 1$ may be trivial if we compare it with the substitutions

$$2\pi \longrightarrow 2\pi i \quad \text{or} \quad 2\pi \longrightarrow i \ .$$

I thought it helpful to use i as unit number for lengths and impulses [2]. Sometimes it may also helpful to describe some parts of physics by using i as unit number of the natural angle unit.

For the following I emphasise the difference of the quaternionic $\boldsymbol{i}$ (with $\boldsymbol{i}^2 = -1$) and the imaginary unit i (with $\mathrm{i} = \sqrt{-1}$) used in [2].

In specialising the vector part of a quaternion to a right-angled isosceles triangle which represents the quaternionic unit $\boldsymbol{i}$ we get by equation (1.2) of [2, page 127] and together with the fundamental metrical equation $ch = i$

$$\boldsymbol{i} = hc^2 j = hc(cj) = hc\mathbf{1} = i\mathbf{1} \tag{11.1}$$

with a vector j right-angled to the triangle, describing the position of this triangle in the 3-dimensional space, with

$$(j\,j) = +1\,, \tag{11.2}$$

where ( ) denotes the scalar product.

And

$$\mathbf{1} := cj\,, \qquad \mathbf{1}^2 := \mathbf{1} \circ \mathbf{1} = +1 \tag{11.3}$$

where $\circ$ denotes the quaternionic product.

I call (11.1)

$$\boldsymbol{i} = i\mathbf{1}$$

the *natural product representation* of the quaternionic unit $\boldsymbol{i}$ by the imaginary unit i and the position $\mathbf{1}$ of the unit $\boldsymbol{i}$. (In my natural metrical system $(c, h) = (1, i)$ i represents Planck's constant, $\mathbf{1}$ Einstein's velocity of light)

Using the function theoretical connections

$$i \cdot \sin\varphi = \sinh(i\varphi)\,, \qquad \cos\varphi = \cosh(i\varphi) \tag{11.4}$$

we reach two parametric descriptions of the characteristic tetra numbers, for example

$$w_{+1} = \sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot \exp\{(\mathbf{1}i)\alpha_1\} = \sinh^{-1}(i\alpha_3) \cdot \sinh(i\alpha_2) \cdot \exp\{\mathbf{1}(i\alpha_1)\}\,. \tag{11.5}$$

I describe the distinction between both formally equivalent parametric representation of w_{+1} in the following way:

Real angles α_κ together with a spin unit $\boldsymbol{i} = i\mathbf{1}$ and circular trigonometrical functions can be used to describe the characteristic number of a tetra.

Alternatively:

Imaginary angles $i\alpha_\kappa$ together with a unit $\mathbf{1}$ and hyperbolic trigonometrical functions can be used to describe the characteristic number of a tetra.

The hyperbolic form

$$W(\alpha_1) = \exp\{\mathbf{1}(i\alpha_1)\} = \cosh(i\alpha_1) + \mathbf{1} \cdot \sinh(i\alpha_1) \tag{11.6}$$

remembers Einstein's description of special relativity. We come a step nearer to this description if we change the angle unit number from 1 to i, so that angles are real, if we use an hyperbolic parametric representation; and angles are imaginary if we use the circular representation.

Such change of the angle unit number does not change the trigonometry of a tetraglobe. Unit numbers as 4, 360, 2π, 1, $2\pi i$ or i, used for describing the natural unit of angles, are only convention.

12. Concluding remarks

Equation (11.6) has the Einsteinean hyperbolic form of the Lorentz operator W. If we represent this operator in the circular form with an imaginary angle unit we have the connection to Minkowski's model: He interpreted Lorentz transformations as Euclidean rotations with imaginary angles. I can only formally, without visual elements realise this concept. But I can visualise W as a triangle with angles measured by an imaginary unit number.

We have a new interesting situation if the trigonometrical restriction $\alpha_\kappa \leq \pi$ is abandoned. With real domain the circular functions possess a periodical, the hyperbolic functions a non-periodical function theoretical continuation if $\alpha_\kappa > \pi$. On the Euclidean level the periodical continuation of the circular functions with real arguments can be illustrated with the help of the movement of a point (with constant velocity) on a circle. A Lorentz transformation can be seen as a physical movement on a straight line with constant velocity (inertial movement) and can be used as a model of the non-periodical continuation of an hyperbolic function. These Euclidean geometrical (physical) models of the analytic continuation of both sets of trigonometric functions are not transferable to the conformal level for on this level the difference of 'circle' and 'straight line' is gone. The '*movement on straight lines*', an axiom of special relativity, is lost on the conformal level. Can we see here a first cause for the experiences that only periodical functions ('waves') can help to describe movements in micro-physics? All conformal lines (circles) are topologically equivalent to Euclidean circles.

This paper gives a new geometrical interpretation of complex numbers as characteristic numbers of tetraglobes. In my next article 'Quaternions as spherical particles of 3-dimensional conformal space' I interpret quaternions as directed and centred tetraglobes. These quaternionic figures, only defined by angles, are the basic elements of a natural 3-dimensional conformal space. The skew field of quaternions in this space of our visual perception must be seen as the *conformal background* of a 'length-metrical' physical world. Only in such world, which is described by units of lengths, space and time are 'separated'. The locality of a 4-dimensional event ('point') in this space can be described if one tetraglobe *which possesses units of lengths* is used as a Cartesian coordinate system. But '*locality*' exists in such space if and only if units of lengths can be defined.

References

[1] E. Cartan, *Les espaces à connexion conforme*, Ann. Soc. Pol. d. math. **2** (1923) 171-221.

[2] K. Ruthenberg, *The Quaternionic Structure of 3-dimensional Natural Geometry*, J. Natural Geom. **16** (1999) 125-140.

[3] K. Ruthenberg, *Measurement of Angles and Elementary Angle Functions defined by Conformal Cross Ratios*, J. Natural Geom. **18** (2000) 131-150. (The misprinted numbering of equations in this article is corrected to the right numbering (2.1) - (9.6) in the reprint, J. Natural. Geom. **19** (2001) 73-92.)

[4] J. A. Schouten, *Ricci-Calculus*, Springer 1954.

[5] T. Y. Thomas, *The differential invariants of generalized spaces*, Cambr.Univ. Press, 1934.

[6] O. Veblen, *Formalism for conformal geometry*, Proc. Nat. Acad. **21** (1935) 168-173.

[7] H. Weyl, *Raum, Zeit, Materie*, Berlin 1923.

[8] K. Yano, *Sur la theorie des espaces a connexion conforme*, J. Imp. Univ. Tokyo **4** (1939) 1-59.

RECEIVED BY THE EDITOR: OCTOBER 9, 2000.